Discrete Geometry Approach to Structure-Preserving Discretization of Port-Hamiltonian Systems

Marko Seslija

Extended Abstract

Computers have emerged as essential tools in the modern scientific analysis and simulation-based design of complex physical systems. The deeply-seated abstraction of continuity immanent to many physical systems inherently clashes with a digital computer’s ability of storing and manipulating only finite sets of numbers. While there has been a number of computational techniques that proposed discretizations of differential equations, the geometric structures they model are often lost in the process. The thesis at hand offers a geometric framework for the discretization of a class of physical systems without destroying the underlying geometric structure of the original system.

Hamiltonian systems are at the foundation of many current physical theories, including quantum and relativistic mechanics, electromagnetism, optics, solid and fluid mechanics. Geometry as the study of observable symmetries and dynamical invariants is de facto the lingua franca of the Hamiltonian theories. The prevailing paradigm in modeling of the complex large-scale physical systems is network modeling. In many problems arising from modern science and engineering, such as multi-body systems, electrical networks and molecular dynamics, the port-based network modeling is a natural strategy of decomposing the overall system into subsystems, which are interconnected to each other through pairs of variables called ports and whose product is the power exchanged between the subsystems.

The formalism that unifies the geometric Hamiltonian and the port-based network modeling is the port-Hamiltonian, which associates with interconnection structure of the network a geometric structure given by a Poisson, or more generally a Dirac structure. The generalized Hamiltonian dynamic is then defined with respect to this Poisson, or Dirac, structure by specifying the Hamiltonian representing the total stored energy, the energy-dissipating elements and the ports of the system. Apart from enunciating a remarkable structural unity, Poisson and Dirac geometry offers a mathematical framework that gives important insights into dynamical systems. Moreover, the geometric formalism transcends the finite-dimensional scenario and has been successfully applied to study of a number of distributed-parameter systems, systems described by a set of partial differential equations.

In this work I provide a framework for structure-preserving discretization of open distributed-parameter systems with generalized Hamiltonian dynamics.

The underlying structure of distributed-parameter systems I address in the thesis is a Stokes-Dirac structure, a type of infinite-dimensional Dirac structure, defined in terms of differential forms on a smooth finite-dimensional orientable, usually Riemannian, manifold with a boundary. The Stokes-Dirac structure generalizes the framework of the Poisson and symplectic structures by providing a theoretical account that permits the inclusion of varying boundary
variables in the boundary problem for partial differential equations. From an interconnection and control viewpoint, such a treatment of boundary conditions is essential for the incorporation of energy exchange through the boundary, since in many applications the interconnection with the environment takes place precisely through the boundary. Well-known examples of distributed-parameter port-Hamiltonian systems on a 3-, 2-, and 1-dimensional manifold, for instance, are: Maxwell’s equations on a bounded domain, the two-dimensional wave equation, and the telegraph equations.

Employing the formalism of discrete exterior calculus, I have introduced simplicial Dirac structures as discrete analogues of the Stokes-Dirac structure and demonstrated that they provide a natural framework for deriving finite-dimensional port-Hamiltonian systems that emulate the behaviors of their infinite-dimensional counterparts. The theory proceeds ab initio by mirroring the continuous setting, and as such is not merely tied to the goal of discretization, but rather aims to offer a geometric framework for the analysis and control of physical systems on discrete manifolds.

The spatial domain, in the continuous theory represented by a finite-dimensional smooth manifold with boundary, is replaced by a homological manifold-like simplicial complex. Familiar examples of such a simplicial manifold are meshes of triangles embedded in the 3-dimensional Euclidian space and tetrahedra obtained by tetrahedrization of a 3-dimensional manifold. The smooth differential forms, in discrete setting, are mirrored by cochains on the primal and dual complexes (Voronoi, i.e., circumcentric dual), while the discrete exterior derivative is defined to be the coboundary operator from algebraic topology. This approach of discrete exterior geometry, rather than discretizing the partial differential equations, allows to first discretize the underlying Stokes-Dirac structure and then to impose the corresponding finite-dimensional port-Hamiltonian dynamics. In this manner, a number of important intrinsically topological and geometric properties of the system are preserved. I demonstrate general considerations on a number of physical examples, including reaction-diffusion systems, where the structure-preserving discretization recovers the standard compartmental model. Furthermore, I show how a Poisson symmetry reduction of Dirac structures associated with infinite- and finite-dimensional models can be conducted in a unified fashion.

In conclusion, this thesis demonstrates how Dirac structures arising from continuous and discrete models can be treated in a unified framework. The consequences of this are that many of the important results from differential geometry can be transferred into the discrete realm and thereby lead to numerically and physically faithful models, which later can be fed to computers and simulate crucial aspects of the physical reality.